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MEASUREMENT OF THE UNCERTAINTY

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Abstract. In recent years, a series of metrics began to develop that allow the quantification of specific properties of process models. These characteristics are, for example, complexity, comprehensibility, maintainability, cohesion and uncertainty. This work is focused on defining a method that allows to measure the uncertainty of process models that was modelled by Stochastic Petri Nets (SPN). The principle of this method consists in mapping the set of all reachable marking of SPN into the continuous-time Markov chain and then calculating its steady-state probabilities. The uncertainty is then measured as the Shannon entropy of the Markov chain (it is possible to calculate the uncertainty of the specific subset of places as well as whole Petri net). Alternatively, the uncertainty is quantified as a percentage of the calculated entropy against maximum entropy.

Keywords: *uncertainty, entropy, modelling, stochastic Petri nets*

1. Introduction and related works

It has long been known that within development, the change of processes are uncertain and interconnected [1-3]. Complexity and uncertainty have become a critical issue for modelling applications, opening new ways for the use and development of models. Increasingly, models are being recognised as essential tools to learn, communicate, explore and resolve the particulars of complex problems [4-6]. However, this shift in the way in which models are used has not always been accompanied by a concomitant shift in the way in which models are conceived and implemented. Too often, models are conceived and built as predictive devices, aimed at capturing best explanations. Considerations of uncertainty are often downplayed and even eliminated because it interfered with the modelling goals. When modelling and analysing business processes, the main emphasis is usually on the validity and accuracy of the model, meaning the model meets the formal specification and also models the correct system. In recent years, a number of measures have begun to develop, enabling quantification of the specific features of process models. These characteristics are, for example, complexity, comprehensibility, maintainability, coherence, and uncertainty. The work is aimed at defining a method that allows one to measure the uncertainty of process models that was modelled using the stochastic Petri nets (SPN). The principle of this method consists of mapping the reachable SPN markings into a continuous Markov chain, and then calculating the stationary probabilities of markings. Uncertainty is then measured as the entropy of the Markov chain (it is possible to calculate the uncertainty of a specific subset of sites as well as the entire network). Alternatively, the uncertainty index is quantified as a percentage of the calculated entropy versus the maximum entropy (the resulting value is normalized to the interval <0.1>). Calculated entropy can also be used as a measure of model complexity [7].

2. Uncertainty

A realistic modelling and simulation of complex systems must include the nondeterministic features of the system and the environment. By 'nondeterministic' we mean that the response of the system is not precisely predictable because of the existence of uncertainty in the system or the environment, or human interactions with the system [8]. Figure 1 shows relationship between uncertainty, data and model.

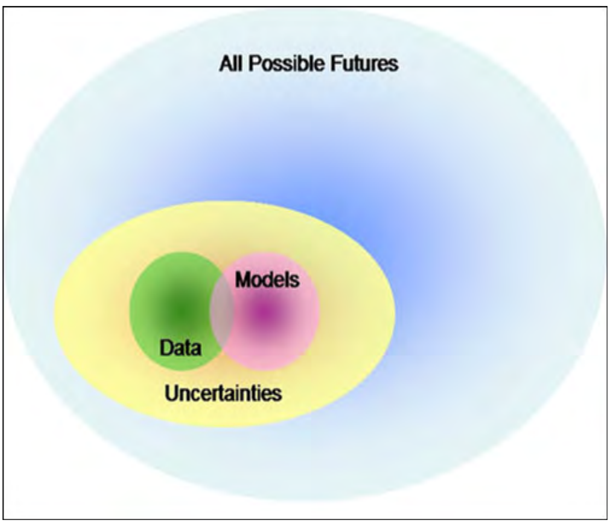


Fig. 1. Uncertainties, Data and Models (according [9])

In a measurement, the uncertainty is quantified as a doubt about the result of the measurement. Measurement device outputs are data displaying information about the measured quantity. In cases where we can use standards to determine instrument accuracy and know the true value of the measured value, we can then identify the measurement error as the difference between the measured value and the true value. In cases where standards do not exist for measured variables, measured results are subject to uncertainty and not to error. We can then use the following relationship for measurement:

$$\text{measurement} = \text{best estimate} \pm \text{uncertainty}$$

Entropy (or uncertainty) and information, are perhaps the most fundamental quantitative measures in cybernetics, extending the more qualitative concepts of variety and constraint to the probabilistic domain. Variety and constraint, the basic concepts of cybernetics, can be measured in a more general form by introducing probabilities. Assume that we do not know the precise states of a system, but only the probability distribution $P(s)$. Variety V can be then expressed as the Shannon entropy H:

$$H(P) = -\sum_{t \in S} P(s) \cdot \log P(s)$$

H reaches its maximum value if all states are equiprobable, that is, if we have no indication whatsoever to assume that one state is more probable than another state. Like variety, H expresses our uncertainty or ignorance about the system's state. It is clear that $H = 0$, if and only if the probability of a certain state is equal to 1 (and all other states are equal to 0). In that case, we have maximal certainty or complete information about what state the system is in. We define constraint that reduces uncertainty, i.e. the difference between maximal and actual uncertainty. This difference can also be interpreted in a different way, for example, as information. Indeed, if we get some information about the state of the system (e.g. through observation), then this will reduce our uncertainty about the system's state by excluding or reducing the probability of a number of states. The information we receive from an observation is equal to the degree to which uncertainty is reduced.

For uncertainty, identification is possible to use the Ishikava fishbone diagram (see Fig. 2).

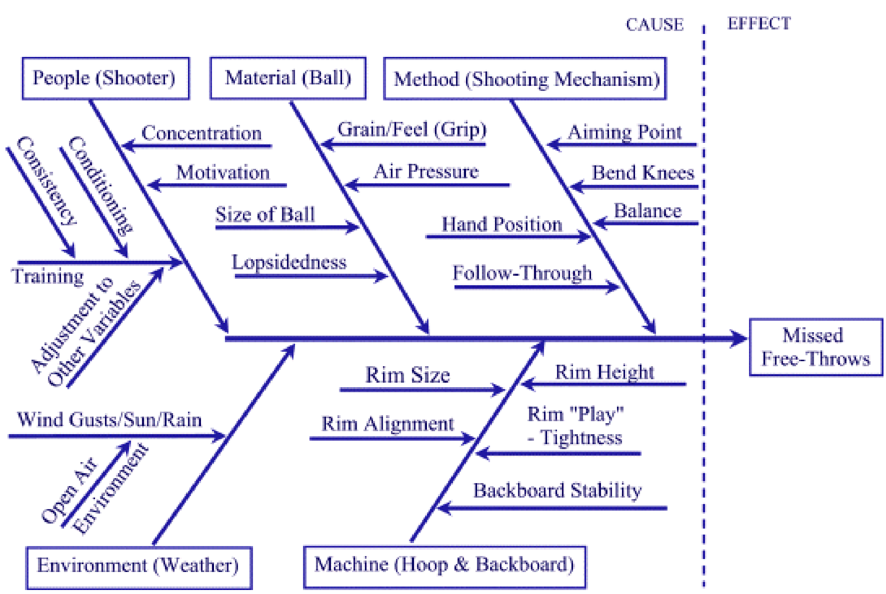


Fig. 2. Fishbone diagram (Source: [11])

Dr. Kaoru Ishikawa developed the "Fishbone Diagram" at the University of Tokyo in 1943. Hence, the Fishbone Diagram is frequently referred to as an "Ishikawa Diagram". The diagram is used in process improvement methods to identify all of the contributing root causes likely to be causing a problem. The Fishbone diagram is an initial step in the screening process. After identifying potential root cause(s), further testing will be necessary to confirm the true root cause(s). This methodology can be used on any type of problem, and can be tailored by the user to fit the circumstances [10]. The example we have chosen to illustrate is "Missed Free Throws" (one team lost an outdoor three-on-three basketball tournament due to missed free throws) [11]. In manufacturing settings, the categories are often: Machine, Method, Materials, Measurement, People, and Environment. In service settings, Machine and Method are often replaced by Policies (high-level decision rules), and Procedures (specific tasks).

3. Petri net

A gentle introduction into the Petri nets modelling approach is made for example by WoPeD [12] where Petri nets are described as follows: "**Petri Nets** are a graphical and mathematical modelling notation first introduced by Carl Adam Petri's dissertation published in 1962 at the Technical University Darmstadt (Germany). A Petri Net consists of **places**, **transitions**, and **arcs** that connect them. Places are drawn as circles, transitions as rectangles and arcs as arrows. Input arcs connect places with transitions, output arcs connect transitions with places. Places are passive components and model the system state. They can contain **tokens**, depicted as black dots or numbers. The current state of the Petri Net (also called the **marking**) is given by the number of tokens at each place. Transitions are active components that model activities that can **occur** and cause a change of the state by a new assignment of tokens to places. Transitions are only allowed to occur if they are **enabled**, meaning that there is at least one token on each input place. By occurring, the transition removes a token from each input place and adds a token to each output place. Due to their graphical nature, Petri Nets can be used as a visualization technique like flow charts or block diagrams but with much more of a scope on concurrency aspects. As a strict mathematical notation, it is possible to apply formal concepts like linear algebraic equations or probability theory for investigating the behaviour of the modelled system. A large number of software tools was developed to apply these techniques.

Examples of properties that are widely verified on Petri's networks are liveness, boundedness, reachability, fairness, and others. Verification of individual properties may be analytical (for basic classes of Petri nets) or have simulation character (for higher classes of Petri nets). The other way of development was to broaden the basic definition of the Petri nets so that their modelling power complies with specific requirements. Examples include timed and stochastic Petri nets, which allow refinement of individual states changes with deterministic [13-15] or stochastic [16] time considerations.

The Stochastic Petri net is a 5-tuple, $SPN = (P, T, F, \Lambda, M_0)$, where:

- $P = \{p_1, p_2, ..., p_m\}$ - a finite set of places,
- $T = \{t_1, t_2, ..., t_n\}$ - a finite set of transitions,
- $P \cap T = \varnothing$ - places and transitions are mutually disjoint sets,
- $F \subseteq (P \times T) \cup (T \times P)$ - a set of edges, defined as a subset of the set of all possible connections,
- $\Lambda : T \to R^+$ - firing rates of exponentially distributed timed transitions,
- $M_0 : P \to N_0$ - an initial marking.

Let $SPN = (P, T, F, \Lambda, M_0)$ be a stochastic Petri net. $M : P \to N_0$, is called marking of *SPN*. Marking of Petri net represents the network state after execution of a specific number of steps, i.e., after firing of a specific number of enabled transitions. Pre-sets and post-sets are defined as:

- ${}^\bullet p = \{t \mid (t, p) \in F\}$ - the pre-set of *p*,
- ${}^\bullet t = \{p \mid (p, t) \in F\}$ - the pre-set of *t*,
- $p^\bullet = \{t \mid (p, t) \in F\}$ - the post-set of *p*,
- $t^\bullet = \{p \mid (t, p) \in F\}$ - the post-set of *t*.

A transition $t \in T$ is called “enabled” within marking *M*, if $\forall p \in {}^\bullet t : M(p) \geq 1$. If a transition $t \in T$ is enabled for the marking *M*, then by execution of *t* is obtained next marking *M’*, which is defined as follows:

$$\forall p \in P : M'(p) = \begin{cases} M(p) - 1, & \text{if } p \in {}^\bullet t \setminus t^\bullet \\ M(p) + 1, & \text{if } p \in t^\bullet \setminus {}^\bullet t \\ M(p) & \text{otherwise} \end{cases}$$

The situation that the transition *t* changes the marking *M* to *M'* is usually expressed as $M[t\rangle M'$. A sequence of transitions σ is the succession of enabled transitions that goes from marking *M* to different marking *M’*. The situation is referred to as $M[\sigma\rangle M'$. A marking is called reachable if there is a sequence of transitions to it from the initial marking. The set of all reachable markings from initial marking M_0 in *SPN* is denoted by $R(M_0)$.

$$R(M_0)=\begin{bmatrix} M_0(p_1) & M_1(p_1) & \cdots & M_{|R(M_0)|}(p_1) \\ M_0(p_2) & M_1(p_2) & \cdots & M_{|R(M_0)|}(p_2) \\ \vdots & \vdots & \ddots & \vdots \\ M_0(p_m) & M_1(p_m) & \cdots & M_{|R(M_0)|}(p_m) \end{bmatrix}$$

Transition rate matrix ***Q*** of *SPN* is defined as $Q:(R(M_0)\times R(M_0))\rightarrow R$. The values are made according to the following rule:

$$\mathrm{Q_{i,j}}=\begin{cases} \sum\limits_{t_k\in\{h:h\in T\wedge M_i\geq 1\wedge M_i[h\rangle M_j\}} \lambda_k, & \text{if } i\neq j \\ -\sum\limits_{M_k\in\{h:h\in R(M_0)\wedge i\neq k\}} \mathrm{Q}_{i,k}, & \text{if } i=j \end{cases}$$

Steady-State distribution vector ***η*** is defined as normalized left null space of transition matrix $Q:\eta Q=0$ and $\eta 1^T=1$.Vector ***η*** represents the steady-state probability of each *SPN* marking:

$$\eta=\begin{bmatrix} Pr(M_0) \\ Pr(M_1) \\ \vdots \\ Pr(M_{|R(M_0)|}) \end{bmatrix}$$

The entropy of the *SPN* is defined as:

$$UncertaintyIndex(SPN)=\frac{H(SPN)}{\log_2|R(M_0)|}$$

4. Example

As an example, a stochastic Petri net is presented consisting of 8 places and 8 transitions (see Fig. 3). The model contains the essential characteristic features that are included in the process model. These elements are, for example, the sequence (e.g., transition T6), AND-split (transition T1), AND-join (transition T8), XOR (transitions T2 and T4) and cycle (transition T8). The presented example can represent an arbitrary business process or workflow.

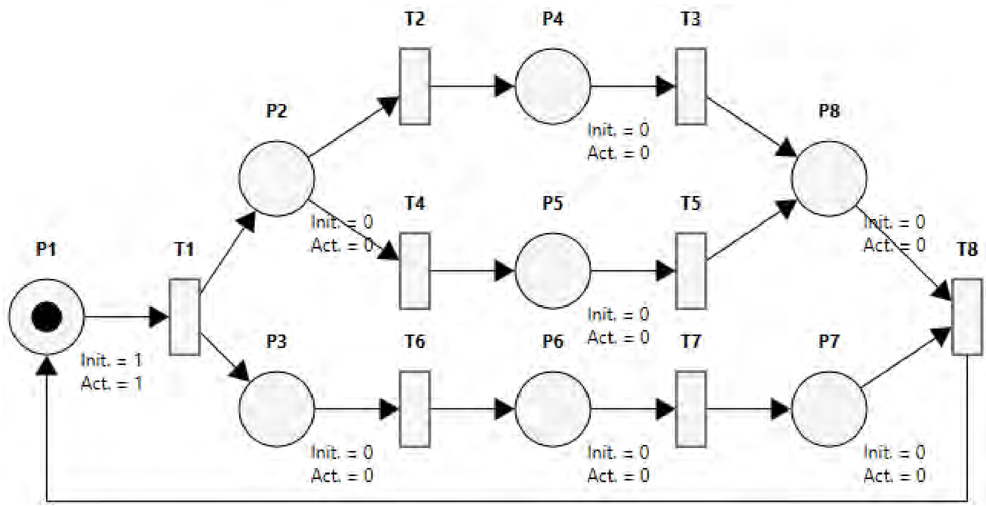


Fig. 3. Example of a stochastic Petri net

The set of all reachable markings $R(M_0)$ of this example Petri net contains 5 markings:

	M_0	M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8	M_9	M_{10}	M_{11}	M_{12}
p_1	1	0	0	0	0	0	0	0	0	0	0	0	0
p_2	0	1	0	0	1	0	0	0	1	0	0	0	0
p_3	0	1	1	1	0	1	0	0	0	0	0	0	0
p_4	0	0	1	0	0	0	1	0	0	0	1	0	0
p_5	0	0	0	1	0	0	0	1	0	0	0	1	0
p_6	0	0	0	0	1	0	1	1	0	1	0	0	0
p_7	0	0	0	0	0	0	0	0	1	0	1	1	1
p_8	0	0	0	0	0	1	0	0	0	1	0	0	1

With consideration of transition firing rates, for example, $\Lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8)$, the given net is shown in Figure 4 as a Markov chain.

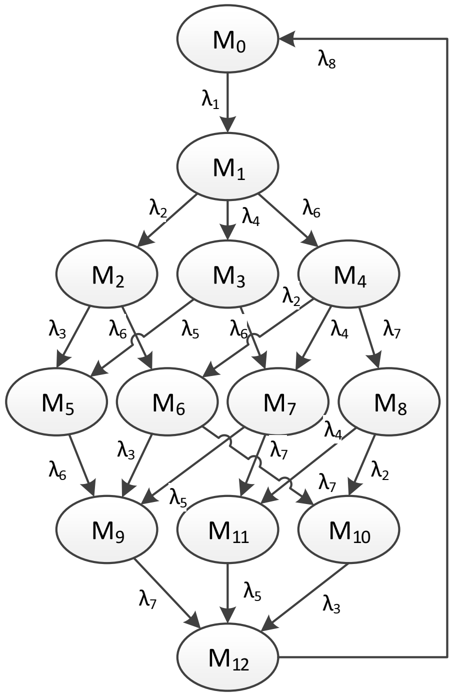


Fig. 4. Corresponding Markov chain

The solution of this chain, for randomly chosen $\Lambda = (7,9,4,2,3,1,8,6)$ is a stationary probability vector as follows:

$$\eta = \begin{bmatrix} 0.0958 \\ 0.0558 \\ 0.1006 \\ 0.0279 \\ 0.0029 \\ 0.4863 \\ 0.0106 \\ 0.0031 \\ 0.0021 \\ 0.0672 \\ 0.0259 \\ 0.0096 \\ 0.1118 \end{bmatrix}$$

Example model with parameters of the exponential distribution is depicted on Figure 5.

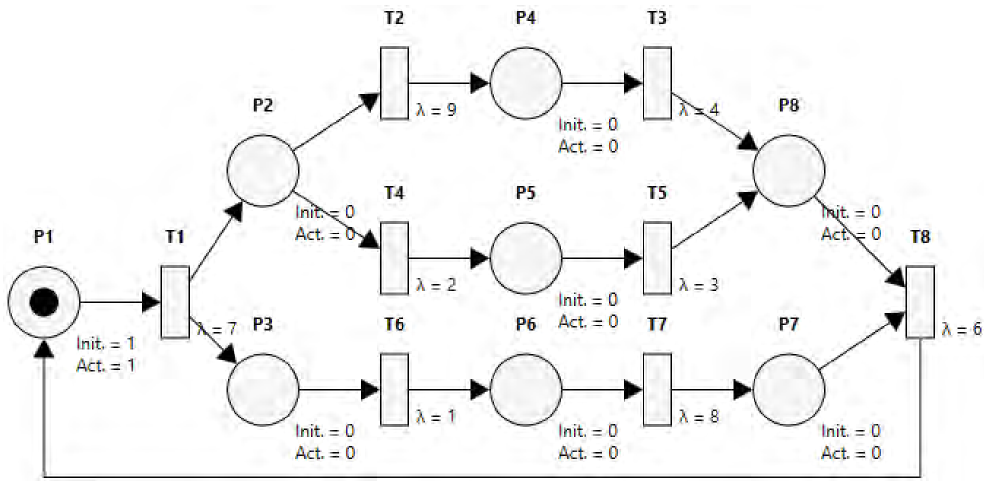


Fig. 5. Example stochastic Petri net with exponential parameters

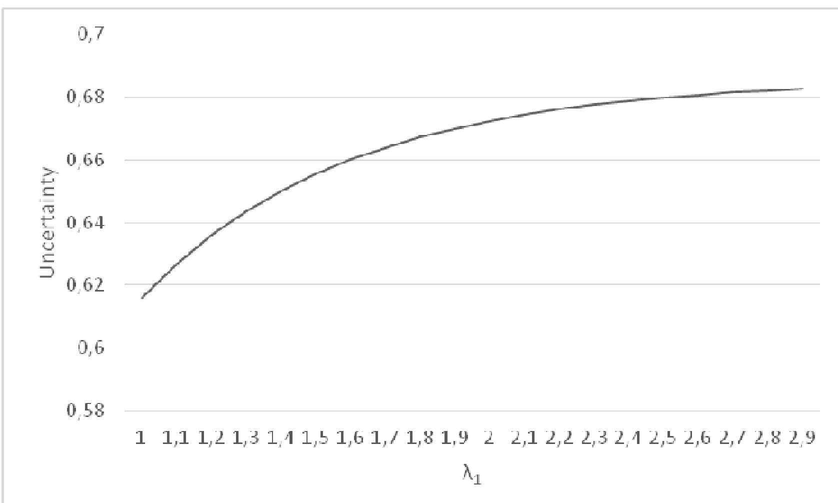


Fig. 6. The development of uncertainty with changing parameter λ_1

The entropy of the example network can then be expressed as follows:

$$H(SPN) = -\sum_{i=1}^{8} \eta_i \log_2 \eta_1 = 2.4987$$

Reference limit (maximum entropy) is in this case is $-\log_2 8 = 3.7004$. The uncertainty for this particular case is determined by the relation $-H(SPN)/\log_2|R(M_0)|$, i.e. $2.4987/3.7004 = 0.6744$. This result can be loosely interpreted as the fact that the uncertainty of the example stochastic Petri net (SPN) reaches 67.44%.

Uncertainty can be then analysed as a response to changes in a parameter of SPN, for example, the number of tokens in the initial marking or an adjustment of a specific parameter $\lambda \in \Lambda$. Figure 6 illustrates the development of uncertainty with changing values of the parameter λ_1.
The shape of the development indicated that with the increasing parameter the uncertainty increases with decreasing momentum. This may mean that the system could become more uncertain and thus the interaction with it is less predictable.

In the following an example is presented that shows the development of the uncertainty with a different initial marking. Figure 7 indicates that the increasing number of tokens in the initial marking (in the place p1) decreases the uncertainty of SPN.

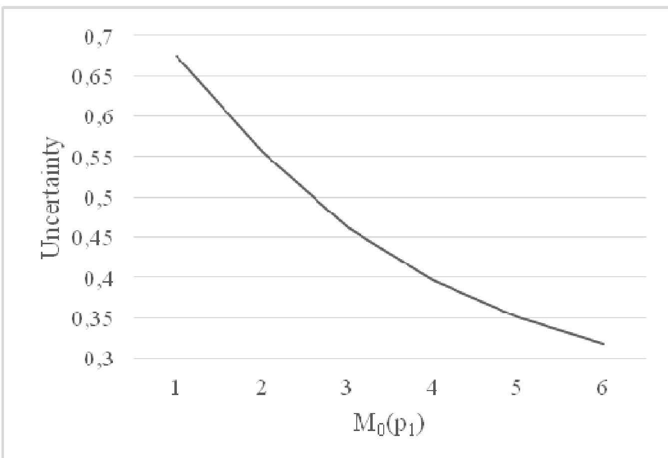


Fig. 7. Uncertainty vs. number of tokens

This development indicates that with an increasing number of tokens in the initial marking (at the place P1), the system become more predictable and thus the work with it is more convenient.

5. Validation

The uncertainty (entropy) of a process can be seen as the complexity of a model, since it reflects properties as the difficulty to understand or maintain it.

There are a number of metrics that have been developed over the last few decades, mainly to measure the complexity of software. Most of these metrics can also be used on the issue of business processes. When defining a new metric it is suitable to perform its validation.

Validation can be of two types, i.e., the theoretical and empirical validation. One of the most widely used theoretical validation measures are Weyuker's properties. The presented metric is validated by Weyuker's properties, which provide an important basis for classifying complexity measures to determine whether they can be qualified as good, structured, and complex. Table 1 contains the fulfilment of all these properties.

Table 1

Fulfilment of Weyuker's properties

	Description	Fulfilment
Property 1	Two different processes should not return the same measurement results	Yes
Property 2	The change in a process should cause a change of its complexity	No
Property 3	It is possible that two distinct processes have the same complexity	Yes
Property 4	A good metric should discriminate different processes (with the same functionality) based on their internal structure (design)	Yes
Property 5	Complexity of the subprocess should be smaller than or equal to the complexity of the original process	No
Property 6	It is possible to have two different processes with the same complexity, but if connected to a third process, their resulting complexities are not equal	Yes
Property 7	Complexity should depend on the order of the statements	Yes
Property 8	Renaming of the process or its components does not change its complexity	Yes
Property 9	It is possible that complexity of two interacting processes is bigger than sum of their individual complexities	Yes

Property 2 is not fulfilled since there may be an infinite number of Petri nets that have the same underlying Markov chain, i.e., the same complexity (entropy). Concerning property 3, if different Petri nets have the same underlying Markov chain, then they have the same complexity (entropy). Property 5 is not fulfilled, since a non-live process, which includes a live subprocess, has the entropy equal to 0, but the complexity of the live subprocess is greater than 0. Property 6 is dependent on the fulfilment of the property 3 and the form of how the two processes are composed together. It is possible to have two different processes with the same complexity, but if connected to a third process, their resulting complexities are not equal. This also depends on the form of how the two processes are composed together.

An example for the Property 6 fulfilment:

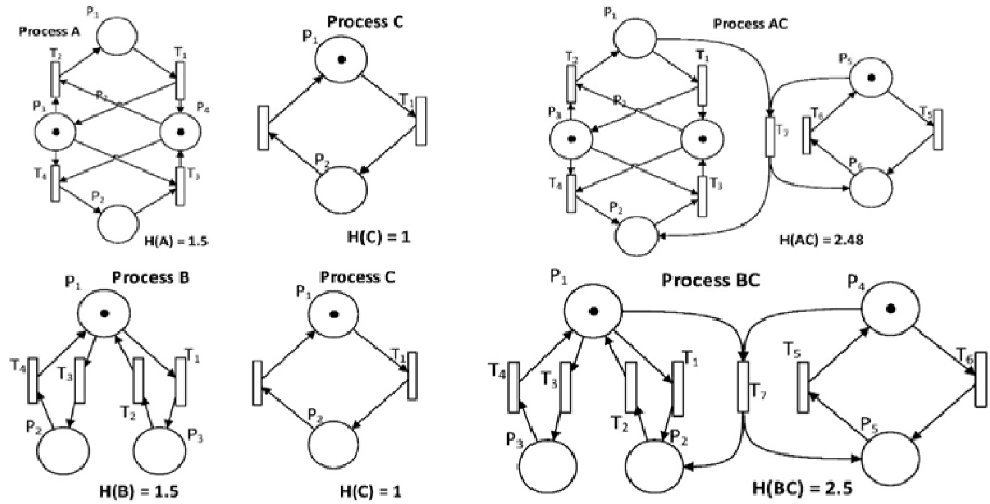


Fig. 8. An example for the Property 6 fulfilment

Property 9 is fulfilled, since a live process is concatenated with a non-live process and the resulting process is live, i.e. the entropy of the resulting process is greater than the sum of individual entropies. An example of the property 9 fulfilment illustrates Figure 9.

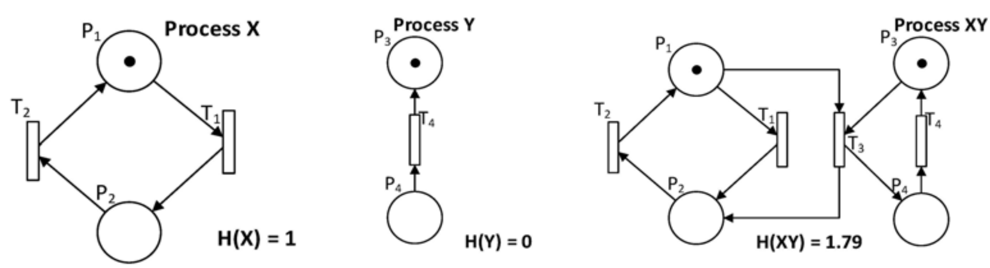


Fig. 9. An example for the Property 9 fulfilment

Further empirical validation was performed (comparison of the existing metrics with our approach).

We have made a comparison of the entropy as a complexity measure with the McCabe's approach and Cardoso's control - flow approach.

McCabe's cyclomatic method (MCM) is defined for each module as $e - n + 2$, where e and n is the number of edges and nodes in the control - flow graph. These charts describe the logical structure of software modules. Nodes represent computational commands or expressions, and the edges represent transmission of control between nodes. Each possible realizable path of the software module has a corresponding path from the input to the output node of the control--flow graph of the module. An important lesson is that the chart of the program is very similar to processes and workflows.

Cardoso's control-flow method (CFM) metric can be used to analyze the complexity of business processes as well as the business processes and processes associated with the website.

Table 2 shows the comparison of the resulting values of the entropy as a complexity measure with our approach, McCabe's approach and Cardoso's approach.

Table 2

The comparison between metrics

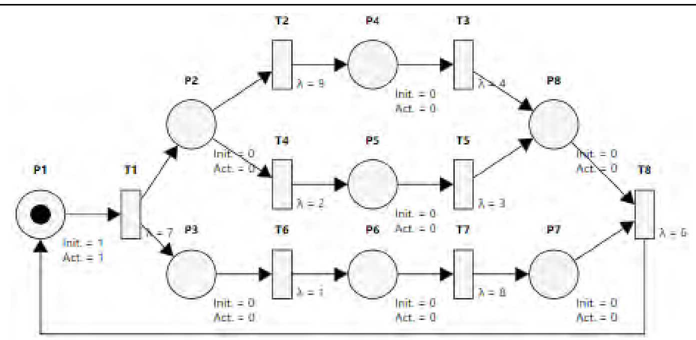
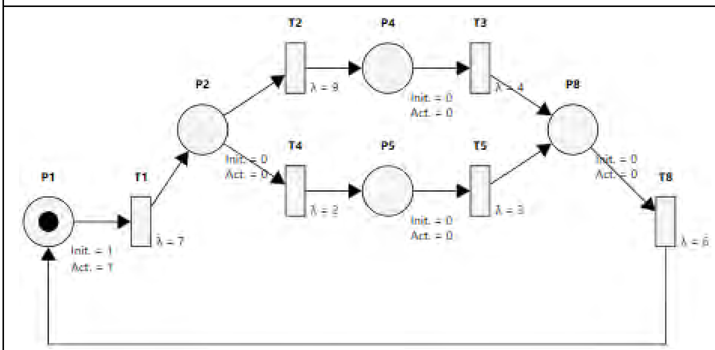
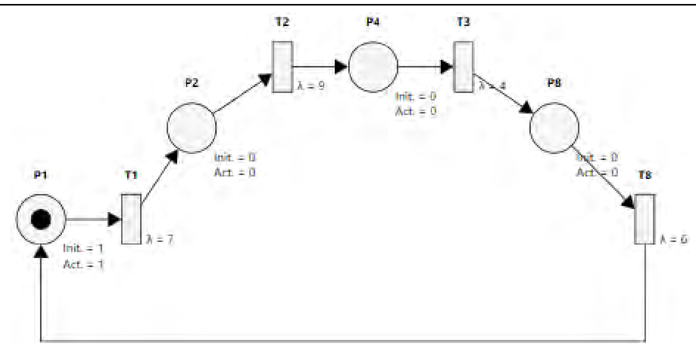
Model	Entropy	MCM	CFM
	2.4957	12	3
	2.2069	3	2
	1.9348	2	0

Table 3 summarizes the compatibility of previously mentioned metrics. The correlation coefficient infers significant dependence between all 3 used methods and thus one can use them interchangeably.

Table 3

Correlations between metrics

		Entropy	MCM	CFM
Entropy	Pearson Correlation	1	.915	.979
	Sig. (2-tailed)		.265	.132
	N	3	3	3
MCM	Pearson Correlation	.915	1	.812
	Sig. (2-tailed)	.265		.396
	N	3	3	3
CFM	Pearson Correlation	.979	.812	1
	Sig. (2-tailed)	.132	.396	
	N	3	3	3

6. Conclusion

Measurement of uncertainty can be an appropriate tool for assessing the relevance and the predictability of process models, and thus serve for more effective managerial decision making. The degree of uncertainty in the process model is directly dependent on two main indicators.

The first is the number, the ratio and the distribution of specific elements (OR, XOR, AND, and LOOP) in the model. These elements provide branching, synchronization and cycles in the model, and are thus the main building blocks of process models that shape its specific structure. One of the other approaches to the measurement of uncertainty in the process model [17] is based on quantifying the entropy of partial substructures of the model at different levels of abstraction. However, this approach takes into account only the static structure of the model and does not take into account the dynamic component, which can be expressed in Petri nets using tokens.

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