THE FOURTH-ORDER EULER ORDINARY DIFFERENTIAL EQUATION WITH THE FRACTIONAL INITIAL/BOUNDARY CONDITIONS

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We consider the initial or boundary-value problem for a fourth-order homogeneous ordinary differential equation with constant coefficients. Detailed solutions to the equation for specific sets of boundary conditions are analyzed. Several detailed solution cases with one or more fractional boundary conditions (described by fractional derivatives) are presented. The final section presents examples that illustrate the solutions to the problem.

Consider a fourth-order differential equation with a constant coefficient, represented by the formula:

$$x^4 \cdot y^{(4)}(x) = a\,y(x) \tag{1}$$

The general solution of equation (1) for a-const is as follows [1]

$$y(x) = C_1 \cdot x^{k_1} + C_2 \cdot x^{k_2} + C_3 \cdot x^{k_3} + C_4 \cdot x^{k_4} \tag{2}$$

where

$$k_{1,2} = \frac{3}{2} \pm \left(\frac{5}{4} + \sqrt{a+1}\right)^{\frac{1}{2}},\ k_{3,4} = \frac{3}{2} \pm \left(\frac{5}{4} - \sqrt{a+1}\right)^{\frac{1}{2}} \tag{3}$$

Example 1: Given fractional order boundary conditions for the appropriate forms, we get:

$$x = x_1 : y(x)\big|_{x=x_1} = B_1, D^{\beta}_{x_2^-} y(x)\big|_{x=x_1} = B_2, x = x_2 : y(x)\big|_{x=x_2} = B_3, D^{\alpha}_{x_1^+} y(x)\big|_{x=x_2} = B_4 \tag{3}$$

where

$$D^{\alpha}_{x_2^-} y(x) := \frac{-1}{\Gamma(1-\alpha)} \int_x^{x_2} \frac{k_i \cdot \xi^{k_i - 1}}{(\xi - x)^{\alpha}} d\xi + \frac{x_2^{k_i}}{\Gamma(1-\alpha)\cdot(x_2 - x)^{\alpha}} \text{, for } 0 < \alpha < 1, i = 1...4 \tag{5}$$

$$D_{x_1^+}^{\beta} y(x) := \frac{1}{\Gamma(1-\beta)} \int_{x_1}^{x} \frac{k_i \cdot \xi^{k_i - 1}}{(x-\xi)^{\beta}} d\xi + \frac{x_1^{k_i}}{\Gamma(1-\beta) \cdot (x - x_1)^{\beta}} \text{, for } 0 < \beta < 1, i = 1...4 \quad (6)$$

where $D_{x_2^-}^{\alpha} y(x)$ and $D_{x_1^+}^{\beta} y(x)$ are the Riemann-Liouville [2] operators.

After introducing the general solution (2) into equations (4), we obtain a system of equations, which can be written in tmatrix form $\mathbf{A} \cdot \mathbf{C} = \mathbf{B}$, where

$$\mathbf{A} = \begin{bmatrix} x_1^{k_1} & x_1^{k_2} & x_1^{k_3} & x_1^{k_4} \\ D_{x_2^-}^{\alpha} x^{k_1}\Big|_{x=x_1^+} & D_{x_2^-}^{\alpha} x^{k_2}\Big|_{x=x_1^+} & D_{x_2^-}^{\alpha} x^{k_3}\Big|_{x=x_1^+} & D_{x_2^-}^{\alpha} x^{k_4}\Big|_{x=x_1^+} \\ x_2^{k_1} & x_2^{k_2} & x_2^{k_3} & x_2^{k_4} \\ D_{x_1^+}^{\beta} x^{k_1}\Big|_{x=x_2^-} & D_{x_1^+}^{\beta} x^{k_2}\Big|_{x=x_2^-} & D_{x_1^+}^{\beta} x^{k_3}\Big|_{x=x_2^-} & D_{x_1^+}^{\beta} x^{k_4}\Big|_{x=x_2^-} \end{bmatrix}, \mathbf{C} = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} \quad (7)$$

Solving the system of equations, we obtain constants C_i, for i =1, 2, 3, 4. Figure 1 shows plots of example particular solutions with the abovementioned boundary conditions.

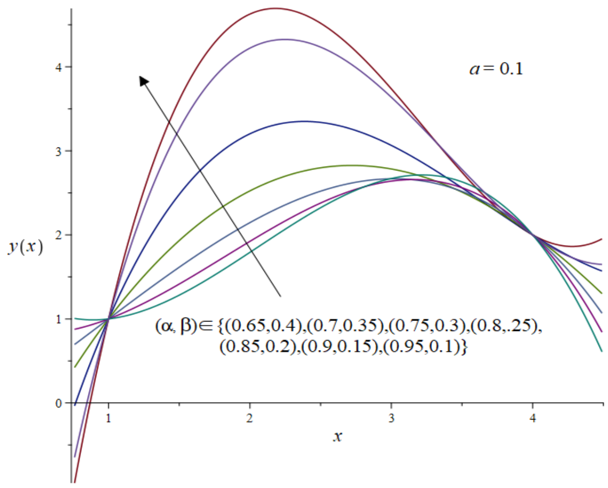

Fig. 1. The solutions of Eq. (1) for boundary conditions $y(x)\big|_{x=1} = 1$, $D_{1^+}^{\alpha} y(x)\big|_{x=4} = 0$, $y(x)\big|_{x=4} = 2$, $D_{4^-}^{\beta} y(x)\big|_{x=1} = -1$.

References

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