# A NEW COHERENCE CONDITION FOR PAIRWISE COMPARISON MATRICES AND ITS RELATIONSHIP TO OTHER ESTABLISHED COHERENCE CONDITIONS

***Tomasz Starczewski*** [1]***, Paweł Tadeusz Kazibudzki*** [2]

[1]*Department of Mathematics, Czestochowa University of Technology, Czestochowa, Poland*
[2]*Department of Marketing Communication, Opole University of Technology, Opole, Poland*
[1] *tomasz.starczewski@pcz.pl,* [2] *p.kazibudzki@po.edu.pl*

Keywords: **analytic hierarchy process, pairwise comparisons, decision making, transitivity, consistency, coherence conditions**

A pairwise comparison matrix (PCM) is a useful tool employed in decision analysis. In multicriteria decision-making support systems, such matrices are used to assess the relative importance of alternatives (or criteria). Based on the decision maker's preferences, the values of the pairwise comparisons are determined [1]. When the decision maker (DM) compares each alternative (or criterion) with the others, a PCM is constructed. In this process, each pair of alternatives is typically compared once in a given order, while the comparison in the reverse order is calculated automatically, as reciprocity of the PCM elements is generally required [2]:

$$a_{ij} = 1/a_{ji} \tag{1}$$

The pairwise comparison method forms the basis of several decision-making support systems, among which the Analytic Hierarchy Process (AHP) is one of the most well-known [3]. However, the usefulness of these methods depends on the quality of the PCMs, which serve as the source of information about the decision maker's preferences [4, 5]. To prevent erroneous information that may arise within the matrices, certain properties are typically required from PCMs. Indeed, since a PCM contains more information than is necessary to estimate the decision maker's priorities, it is possible to detect inconsistencies in the decision maker's judgments [6].

However, even when a matrix exhibits inconsistencies, identifying their exact source can be difficult. It is only occasionally possible to pinpoint a single matrix element that disrupts its coherence. More often, PCM inconsistency results from dispersed inaccuracies in estimating priority ratios. In some cases, the inconsistency may also be linked to the procedure itself, as the values used in PCMs are typically limited to the nine natural numbers and their reciprocals [7, 8]. Consequently, attempts to correct the matrix may worsen its quality [5], and the only reasonable solution is to ask the decision maker to revise the matrix.

However, in some cases, the inconsistency is not significant, and the priorities derived from the PCM may still be sufficiently reliable for the decision maker's purposes [3, 8]. In such situations, it is important to establish reasonable conditions under which a PCM can serve as a useful source of information for estimating priorities. At the same time, these conditions should not overly constrain the decision maker's judgments. For this reason, a variety of consistency conditions have been proposed in the literature [9, 10].

Since the inception of the AHP, a condition used to verify the quality of PCMs has been consistency (sometimes referred to as cardinal transitivity) [11]. The PCM $(n \times n)$ satisfies this condition, if all its elements subject to the following equality (for $i, j, k = 1, \dots, n$):

$$a_{ij} = a_{ik} a_{kj} \tag{2}$$

Indeed, when a PCM satisfies condition (2), the decision maker's priority vector can be readily derived [1]. However, in practice, PCMs typically do not meet the consistency condition [2]. Consequently, alternative conditions for PCMs have been proposed. One of the most intuitive is the (ordinal) transitivity condition. This condition is well established in normative decision-making theory and is regarded as one of the fundamental axioms that a rational decision maker should satisfy [10]. For example, it is one of the two conditions required to define the utility function developed by von Neumann and Morgenstern [12]. In practice, most decision makers' judgments conform to this axiom. In the context of PCMs, this condition implies that the elements of a reciprocal matrix follow the rule:

$$a_{ik} \geq 1 \ \wedge \ a_{kj} \geq 1 \ \Longrightarrow \ a_{ij} \geq 1 \tag{3}$$

Six additional coherence conditions have been identified and investigated by Brunelli and Costa [8, 9]. Four of these – max–max transitivity, weak consistency, index exchangeability, and quasi-consistency – lie between the transitivity and consistency conditions. This means that all matrices satisfying these conditions are transitive, while consistent matrices satisfy each of them. Moreover, these four conditions form a nested sequence of matrix subsets.

The remaining two conditions, pairwise dominance and rank order preservation, define sets that partially overlap with the set of transitive matrices. When indifference is not considered, these sets coincide with the set of max–max transitive matrices [10].

Although the coherence conditions discussed in the literature provide a broad classification framework for PCMs, which is sufficient for many users of AHP and other pairwise comparison methods, additional conditions can also be developed. One such condition can be defined for reciprocal matrices as follows:

$$\min_{i<k<j}\left\{a_{ik},\ a_{kj},\ a_{ik} \cdot a_{kj}\right\} \leq \ a_{ij} \ \leq \max_{i<k<j}\left\{a_{ik}\ ,\ a_{kj},\ a_{ik} \cdot a_{kj}\right\} \tag{4}$$

This condition is closely related to the transitivity and consistency conditions. It is evident that matrices satisfying condition (4) are transitive, and that consistent matrices also satisfy condition (4). Therefore, this new condition can be regarded as lying between transitivity and consistency. Accordingly, this coherence condition may be referred to as "consistency-driven transitivity."

Although its relationship to the transitivity and consistency conditions is quite evident, the connection between this new condition and other coherence conditions is less clear. Therefore, in our study, we examine different types of matrices to determine which satisfy this new condition and which do not. We anticipate that this analysis will contribute to a better understanding of the relationships between this class of "consistency-driven" matrices and those that satisfy other coherence conditions. We also suggest that this condition may serve as an additional criterion for assessing the usefulness of PCMs.

## References


[1] Brunelli M., Introduction to the Analytic Hierarchy Process, SpringerBriefs in Operations Research, Springer Cham 2015.

[2] Grzybowski A.Z., Note on a new optimization based approach for estimating priority weights and related consistency index, Expert Systems with Applications 2012, 39 (14), 11699-11708.

[3] Ishizaka A., Siraj S., Are multi-criteria decision-making tools useful? An experimental comparative study of three methods, European Journal of Operational Research 2018, 264(2), 462-471.

[4] Grzybowski A.Z., Starczewski T., New look at the inconsistency analysis in the pairwise-comparisons-based prioritization problems, Expert Systems with Applications 2020, 159, 113549.

[5] Kazibudzki P.T., The uncertainty related to the inexactitude of prioritization based on consistent pairwise comparisons. PLoS ONE 2023, 18(9): e0290751. https://doi.org/10.1371/journal.pone.0290751

[6] Ishizaka A., Siraj S., Interactive consistency correction in the analytic hierarchy process to preserve ranks, Decisions in Economics and Finance 2020, 43, 443–464.

[7] Cavallo B., Ishizaka A., Evaluating scales for pairwise comparisons, Annals of Operations Research 2023, 325, 951–965.

[8] Cavallo B., Ishizaka A., Olivieri M.G., Squillante M., Comparing inconsistency of pairwise comparison matrices depending on entries, Journal of the Operational Research Society 2019, 70(5), 842–850.

[9] Brunelli M., Cavallo B., Incoherence measures and relations between coherence conditions for pairwise comparisons, Decisions in Economics and Finance 2020, 43, 613–635.

[10] Brunelli M., Cavallo B., Distance-based measures of incoherence for pairwise comparisons, Knowledge-Based Systems 2020, 187, 104808.

[11] Saaty T.L., The Analytic Hierarchy Process, McGraw Hill, New York 1980.

[12] Von Neumann J., Morgenstern O., Theory of games and economic behavior, Princeton University Press, Princeton, 1944.