

## APPROXIMATION OF DIRECTIONAL FRACTIONAL OPERATORS ON GFDM STENCILS USING LEARNED KERNEL BASES

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The paper presents a discrete approximation of directional fractional operators on irregular stencils of the generalized finite difference method (GFDM). The operator is applied to local classical derivative fields  $Q \in \{T_x, T_y, T_{xx}, T_{xy}, T_{yy}\}$  determined in the neighbourhood of the central point  $\mathbf{x}_c$ . The direction of nonlocal spatial memory is defined by the angle  $\phi$  and the unit vector  $\mathbf{e}_\phi = (\cos \phi, \sin \phi)$ . This representation describes interactions aligned with a selected axis of propagation, transport, or diffusion. For neighbours  $\mathbf{x}_i \in \mathcal{S}(\mathbf{x}_c)$  the operator is written as a weighted sum of centre-neighbour increments

$$D_\phi^\alpha Q(\mathbf{x}_c) = \frac{\sum_{\mathbf{x}_i \in \mathcal{S}(\mathbf{x}_c)} w_i^{(\alpha, \phi)} (Q(\mathbf{x}_c) - Q(\mathbf{x}_i))}{\sum_{\mathbf{x}_i \in \mathcal{S}(\mathbf{x}_c)} w_i^{(\alpha, \phi)} + \varepsilon}, \quad (1)$$

where  $\alpha$  is the order of the operator. The weights  $w_i^{(\alpha, \phi)}$  depend on the radial distance, the projection onto the direction  $\phi$ , and the angular agreement between the neighbour position and the memory axis. The construction gives zero response for constant fields and yields a finite directional representation of fractional memory on irregular geometries.

The learned part does not replace the operator structure. It approximates its kernel by a compact basis representation. The basis block generates six kernel distributions on the same stencil: forward, backward, radial, angular, and two transverse distributions. After normalization over the active neighbours, the bases  $K_b(i)$  are contracted with differences of the classical derivatives  $dT_x, dT_y, dT_{xx}, dT_{xy}$  and  $dT_{yy}$ . The resulting 30 operator features are used by the decoder to predict five components:  $D_\phi^\alpha T_x, D_\phi^\alpha T_y, D_\phi^\alpha T_{xx}, D_\phi^\alpha T_{xy}$  and  $D_\phi^\alpha T_{yy}$ . The first-order components

also form an intermediate information structure for the reconstruction of the second-order components.

Numerical experiments were carried out on irregular  $50 \times 50$  grids, 25-point stencils, and second-order GFDM derivatives. The study included the orders  $\alpha = 0.1, 0.2, \dots, 2.0$  and directions parameterized by  $\phi$ . On the test set, the method achieved  $R^2 = 0.9531 - 0.9705$  and  $\text{rRMSE} = 0.1717 - 0.2164$ . In the out-of-distribution test, it achieved  $R^2 = 0.9083 - 0.9430$  and  $\text{rRMSE} = 0.2387 - 0.3029$ . The results show that learned kernel bases provide a compact, interpretable, and stable representation of directional fractional operators on irregular grids.

## References

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