A PHYSICS-INFORMED NEURAL SURROGATE FOR THE CAPUTO FRACTIONAL DERIVATIVE

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Fractional calculus extends the classical concepts of differentiation and integration to non-integer orders and has become a valuable tool for modeling complex phenomena in fields such as mechanics, biology, economics, and chemistry. Its effectiveness stems from the nonlocal nature of fractional operators, which allows for the accurate description of memory effects and long-range interactions that are not captured by standard integer-order models. However, this nonlocality also leads to significant mathematical and computational challenges, often making analytical solutions difficult to obtain. For this reason, the development of efficient approximation methods is of central importance in both theoretical and applied research. In recent years, machine learning techniques, and in particular neural networks, have gained attention as powerful tools for approximating complex operators.

In this work, we propose a neural surrogate model for approximating the left Caputo fractional derivative, defined as [1]

$$ {}^{C}D_{0^+}^{\alpha} y(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-s)^{-\alpha} y'(s)\,ds, \qquad \alpha \in (0,1). \tag{1} $$

Our approach incorporates the history-dependent nature of fractional operators into the model architecture, aligning it with the mathematical structure of the Caputo derivative. The model is evaluated on both in-distribution and out-of-distribution datasets, demonstrating good generalization beyond the training domain. Furthermore, the effectiveness of the surrogate is validated on representative fractional differential equations of the form [2]

$$ {}^{C}D_{0^+}^{\alpha} y(x) - \lambda y(x) = 0, \tag{2} $$

with behavior depending on the order α:

- for $0 < \alpha < 1$, the equation describes a fractional relaxation process,
- for $1 < \alpha < 2$, the solution exhibits oscillatory dynamics.

The results demonstrate that the proposed approach provides an effective framework for learning fractional operators, with potential applications in scientific computing and modeling of real-world phenomena.

References

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